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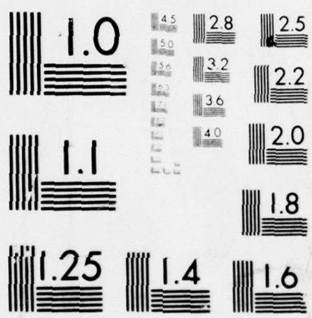

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The problem of the sliding of a rigid asperity over a power law visco-elastic layer is examined in the realistic limit of infinite (dimensionless) layer thickness. For a contact interval of unit length, asymptotic expansions for the normal traction over the interval together with several other physically relevant quantities (e.g. the friction coefficient) are developed in term of an appropriate asymptotic sequence of powers of the (dimensionless) layer thickness.

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Summary

The problem of the sliding of a rigid asperity over a power law visco-elastic layer is examined in the realistic limit of infinite (dimensionless) layer thickness. For a contact interval of unit length, asymptotic expansions for the normal traction over the interval together with several other physically relevant quantities (e.g. the friction coefficient) are developed in terms of an appropriate asymptotic sequence of powers of the (dimensionless) layer thickness.

Introduction

In a previous paper by Walton, Nachman and Schapery [1], henceforth referred to as I, the 2-D problem of the sliding of a rigid asperity over a power law viscoelastic halfspace was considered. Here a power law viscoelastic material is defined as one whose stress-strain relation is

$$\sigma_{ij} = \int_{-\infty}^{t} G_{ijk\ell}(t-\tau)\frac{\partial\varepsilon_{k}}{\partial\tau}\, d\tau$$

where the compliance moduli, $G_{ijk\ell}$, are given by

$$G_{ijk\ell}(t) = A_{ijk\ell}\, t^{-\alpha} H(t), \quad 0 < \alpha < 1$$

and $H(t)$ is the Heaviside function. The rheological arguments favoring such a model may be found in [2] and references therein.

The analogous problem for a finite layer is much more difficult. So much so that when Alblas and Kuipers [3] attempted it for an elastic layer they contented themselves with a thick layer asymptotics approach. In the spirit of their work we present a similar investigation for a power law viscoelastic layer.

1. Formulation of the Problem

We now consider the problem of the steady, frictionless sliding to the left, with velocity U, of a rigid asperity over the power law viscoelastic layer, $0 \le y \le \theta$. The layer is rigidly constrained at $y = \theta$.

Neglecting inertia, the force balance equations are

$$\sigma_{ij,j} = 0$$

and the boundary conditions are

$$\begin{array}{ll} \sigma_{12}(x,0,t) = 0 & -\infty < x < \infty \\ v(x,0,t) = f(x + Ut) & a(t) < x < b(t) \\ \sigma_{22}(x,0,t) = 0 & x < a(t),\ x > b(t) \\ v(x,\theta,t) = u(x,\theta,t) = 0 & -\infty < x < \infty \\ a(t) = a_0 + Ut, & \\ b(t) = b_0 + Ut & \end{array}$$

Here $v(x,y,t)$ and $u(x,y,t)$ are the vertical and horizontal displacements and $f(x)$ is the shape of the indentor.

We choose units so that $a_0 = 0$ and $b_0 = 1$ and thus all length quantities may be henceforth considered dimensionless. We also adopt the Gallilean variable $s = x + Ut$.

The next series of steps is more or less standard and the details may be found in I. The force balance equations and boundary conditions are Fourier transformed with respect to s, the resulting ODE's in y are solved and finally a Fourier inversion results in the following integral equation relating the displacement $f(s)$ to a dimensionless normal traction $g(s)$ $(\propto \sigma_{22}(s,0))$:

$$2\pi f(s) = \int_{-\infty}^{\infty} dp\ \Phi(p\theta)\frac{(ip)^{-\alpha}}{|p|}\ e^{ips}\int_0^1 g(\xi)e^{-ip\xi}d\xi \tag{1}$$

$$\Phi(p\theta) = \frac{2|p\theta| - (3-4\tilde{\nu})\ \sinh 2|p\theta|}{2p^2\theta^2 + (3-4\tilde{\nu})\ \cosh 2p\theta + (5-12\tilde{\nu}+8\tilde{\nu}^2)}$$

where $\tilde{\nu} = ip\bar{\nu}$ and $\bar{\nu}$ is the Fourier transform of Poisson's ratio ν. For a wide variety of viscoelastic materials $\tilde{\nu}$ is a constant [2]. The analysis of (1) is the major thrust of this investigation. It is important to remember that (1) is valid for all s. However, $f(s)$ is only given for $0 \le s \le 1$. Thus once we find $g(s)$ for $0 \le s \le 1$ we use (1) to define $f(s)$ everywhere. Moreover, it is easy to show a priori that $f(s) \sim |s|^{-\alpha}$ for $|s| >> 1$.

2. Analysis of Thick Layer Problem

For the sake of algebraic convenience we take $\tilde{\nu} = 1/2$ and set $z = p\theta$. Then (1) becomes, after differentiation,

$$\pi f'(s) = -\theta^{\alpha-1} \int_0^1 g(\xi)d\xi \int_0^\infty z^{-\alpha}\left(\frac{\sinh z \cosh z - z}{\cosh^2 z + z^2}\right) \cdot \sin\left[\frac{z}{\theta}(s-\xi) - \frac{\alpha\pi}{2}\right] dz \tag{2}$$

or

$$\begin{aligned}\pi f'(s) = &-\Gamma(1-\alpha)\left[\cos\alpha\pi \int_0^s \frac{g(\xi)d\xi}{(s-\xi)^{1-\alpha}} - \int_s^1 \frac{g(\xi)d\xi}{(\xi-s)^{1-\alpha}}\right] \\ &- \theta^{\alpha-1}\int_0^1 g(\xi)d\xi \int_0^\infty z^{-\alpha}\left(\frac{\sinh z \cosh z - z}{\cosh^2 z + z^2} - 1\right) \\ &\sin\left[\frac{z}{\theta}(s-\xi) - \frac{\alpha\pi}{2}\right]dz .\end{aligned} \tag{3}$$

Since $\frac{\sinh z \cosh z - z}{\cosh^2 z + z^2} - 1$ is exponentially small for large z one can show that

$$\int_0^\infty z^{-\alpha}\left(\frac{\sinh z \cosh z - z}{\cosh^2 z + z^2} - 1\right)\sin\left[\frac{z}{\theta}(s-\xi) - \frac{\alpha\pi}{2}\right]dz$$

$$= \sum_{n=0}^{\infty} \frac{a_n}{\theta^n n!}(s-\xi)^n$$

where

$$a_n = (-1)^{n+1}\sin\left[(\alpha+n)\frac{\pi}{2}\right]\int_0^\infty z^{n-\alpha}\left(\frac{\sinh z \cosh z - z}{\cosh^2 z + z^2} - 1\right)dz.$$

See Table 1.

Note that for large n the integral above is $\sim n!/4^{n+1}$. In any event we may write (3) as

$$\text{(4)} \qquad \pi f'(s) = -\Gamma(1-\alpha)\left[\cos\alpha\pi \int_0^s \frac{g(\xi)d\xi}{(s-\xi)^{1-\alpha}} - \int_s^1 \frac{g(\xi)d\xi}{(\xi-s)^{1-\alpha}}\right]$$

$$- \sum_{n=0}^{\infty} \frac{a_n}{\theta^{n+1-\alpha} n!} \int_0^1 g(\xi)(s-\xi)^n d\xi .$$

To "uncouple" equation (4) we assume $\theta >> 1$ and seek asymptotic expansions for $f'(s)$ and $g(s)$ in terms of the gauge functions $1, \varepsilon^{p_1}, \varepsilon^{p_2}, \ldots$ $(\varepsilon = \theta^{-1})$ where $p_n = n - \alpha$. Thus

$$\text{(5)}_0 \qquad f_0'(s) = \frac{\Gamma(1-\alpha)}{\pi}\left[\int_s^1 \frac{g_0(\xi)d\xi}{(\xi-s)^{1-\alpha}} - \cos\alpha\pi \int_0^s \frac{g_0(\xi)d\xi}{(s-\xi)^{1-\alpha}}\right]$$

$$\text{(5)}_1 \qquad f_1'(s) + a_0 \int_0^1 g_0(\xi)d\xi = \frac{\Gamma(1-\alpha)}{\pi}\left[\int_s^1 \frac{g_1(\xi)d\xi}{(\xi-s)^{1-\alpha}} - \cos\alpha\pi \int_0^s \frac{g_1(\xi)d\xi}{(s-\xi)^{1-\alpha}}\right]$$

etc.

In each case the right hand side is always the same generalized Abel integral operator and the left hand side is known for $0 \leq s \leq 1$.

Equation (5) has been extensively analysed in I for a variety of asperity shapes and so for the purposes of illustration we consider the problem for a cylindrical indentor of radius R:

$$f(s) = -\frac{(s-c)^2}{2R} + d, \quad 0 \leq s \leq 1.$$

Having chosen an asperity shape the issue of principal interest now is the behavior of $g_0(s)$ as a function of α. The solution exhibits markedly different behavior for $\alpha < 1/2$, $\alpha = 1/2$ and $\alpha > 1/2$. In particular, continuity of the normal traction is absent for $\alpha \geq 1/2$ and, indeed, for $\alpha > 1/2$ the traction possesses an integrable singularity at the leading edge, $s = 0$. Of course for $\alpha = 0$ all the classical results from elasticity are recovered, in particular the results of Alblas and Kuipers [3].

The solutions were derived in I and we catalog them below.

$\alpha < 1/2$:

(6) $$g_0(s) = \frac{R^{\alpha-1}}{\Gamma(1-\alpha)} \int_s^1 \xi^{\frac{1}{2}-\alpha} (1-\xi)^{-1/2}\, d\xi$$

$$- \frac{R^{\alpha-1}\Gamma(1/2)(\frac{1}{2}-\alpha)}{\Gamma(1-\alpha)\Gamma(2-\alpha)\Gamma(\alpha+\frac{1}{2})} \int_s^1 \frac{\xi^{-1/2}(1-\xi)^{\alpha-\frac{1}{2}}}{(\xi-s)^{\alpha}}\, d\xi$$

where the auxilliary condition

(7) $$\int_0^1 (\xi-c_0)\xi^{-1/2}(1-\xi)^{-\alpha-\frac{1}{2}}\, d\xi = 0$$

must be enforced. Clearly equation (7) defines c_0, where c_0 is the first approximation to c, the apex of the indentor. This $c_0 = 1/2(1-\alpha)$ which, for $0 < \alpha < 1/2$, gives a value between 0 and 1. The function $g_0(s)$ is continuous on $[0,1]$. See Figure 1.

$\alpha = 1/2$:

This is the easiest case mathematically and gives

(8) $$g_0(s) = 2[(1-s)/\pi R]^{1/2}$$

with $c_0 = 1$. Thus the material leaves at the apex of the indentor. Note that (8) predicts discontinuous behavior of $g_0(0)$. See Figure 1.

$\alpha > 1/2$:

The expression for $g_0(s)$ given by (6) is equally valid for this range of α as is the prescription $c_0 = 1/2(1-\alpha)$.

Note, however, that now $c_0 > 1$ so that the material leaves before the apex. Moreover, $g_0(s)$ is now singular at the leading edge. Indeed,

(9) $$g_0(s) = \begin{cases} O(1-s)^{1/2} & s \to 1- \\ O(s)^{\frac{1}{2}-\alpha} & s \to 0+ \end{cases}$$

for all α and is thus singular for $\alpha > 1/2$. See Figure 1.

Two physically meaningful quantities associated with this problem are the total load $(P = \int_0^1 \sigma_{22} ds)$ and the friction coefficient $(C_f = \int_0^1 f'\sigma_{22} ds / \int_0^1 \sigma_{22} ds)$.

The preceding work allows us to exhibit the first approximations to both of these quantities,

(10) $$P_0 = \frac{\Gamma(3/2)\Gamma(\frac{3}{2} - \alpha)R^{\alpha-2}}{(1-\alpha)\Gamma(3-\alpha)}$$

(11) $$C_f^0 = \alpha(2-\alpha)/R(1-\alpha)(3-\alpha).$$

The one remaining unknown quantity is d, the depth of penetration at $x = 0$. To facilitate the computation of d we rewrite (1) as follows,

$$\pi f(s) = \theta^{\alpha} \int_0^{\infty} z^{-\alpha-1} \phi(z) \cos\left(\frac{zs}{\theta} - \frac{\alpha\pi}{2}\right) dz \int_0^1 g(\xi) d\xi$$

$$+ \frac{\Gamma(1-\alpha)}{\alpha} \left[s^{\alpha} \cos \alpha\pi \int_0^1 g(\xi) d\xi - \cos \alpha\pi \int_0^s g(\xi)(s-\xi)^{\alpha} d\xi \right.$$

$$\left. - \int_s^1 g(\xi)(\xi-s)^{\alpha} d\xi \right] + \theta^{\alpha-1} \int_0^1 g(\xi) d\xi \int_0^{\xi} dy \int_0^{\infty} z^{-\alpha}$$

$$\cdot [1 - \phi(z)] \sin\left(\frac{z(y-s)}{\theta} + \frac{\alpha\pi}{2}\right) dz .$$

Clearly d_0 will be an $O(\theta^{\alpha})$ quantity, a result which could also have been predicted from I as it was shown there that the displacement was $O(s^{\alpha})$ for large s. This is analogous to the results of Alblas and Kuipers [3] who had an $O(\ln \theta)$ penetration depth reflecting the well known fact that for an elastic halfspace the displacement is $O(\ln s)$ for large s.

In any event, we have

$$d_0 = \frac{P_0 \theta^{\alpha}}{\pi} \int_0^{\infty} z^{-\alpha-1} \phi(z) dz + O(1) \tag{12}$$

and values of the above integral for various values of α will be provided below.

Successive terms in the asymptotic expansion are deduced exactly as the lowest order quantities were. For example,

$$c_1/R + a_0 P_0 = \frac{\Gamma(1-\alpha)}{\pi} \left[\int_s^1 \frac{g_1(\xi) d\xi}{(\xi-s)^{1-\alpha}} - \cos \alpha\pi \int_0^s \frac{g_1(\xi) d\xi}{(s-\xi)^{1-\alpha}} \right] \tag{13}$$

serves to define both $g_1(s)$ and c_1. In this case no bounded solution exists unless

$$c_1 = -a_0 P_0 R \quad \text{and then} \quad g_1(s) = 0.$$

It thus follows that $P_1 = 0$, and $C_f^1 = -a_0 P_0$.

Proceeding to the next order gives

$$\text{(14)} \quad c_2/R + a_1 \int_0^1 g_0(\xi)(s-\xi)d\xi = \frac{\Gamma(1-\alpha)}{\pi}\left[\int_s^1 \frac{g_2(\xi)d\xi}{(\xi-s)^{1-\alpha}} - \cos\alpha\pi \int_0^s \frac{g_2(\xi)d\xi}{(s-\xi)^{1-\alpha}}\right].$$

The solvability condition for this problem defines c_2 as

$$\text{(15)} \quad c_2 = \frac{Ra_1}{4(1-\alpha)}\left[\int_0^1 \xi g_0(\xi)d\xi - P_0\right] = \frac{Ra_1}{4(1-\alpha)}\{RC_f^0 + (c_0 - 1)P_0\}.$$

The traction $g_2(s)$ may be found from equations (6), (8) by the simple expedient of replacing R by 1 and $g_0(s)$ by $-g_2(s)/a_1P_0$. Clearly for any value of α, $g_2(s)$ enjoys all the properties that $g_0(s)$ had.

We can also see from this last result that

$$P_2 = -1/a_1 \quad .$$

From the representative table supplied below it is clear, for example, that $c_1 < 0$, $C_f^1 < 0$, $c_2 > 0$ and $P_2 > 0$. Thus the apex is shifted to the left by the finiteness of the layer, which is what one would expect. Moreover, since the material ultimately returns to its original position the friction coefficient is lowered.

Summary

Using the closed form solutions for the sliding of a rigid asperity over a power law viscoelastic halfspace we are able to supply, in terms of an asymptotic representation, corresponding quantities for the case of a large finite layer.

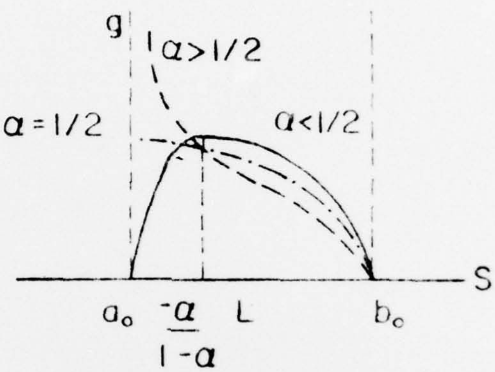
$\alpha \backslash n$	0	1	2	3	4
1/4	0.76	-1.72	-1.217	6.799	8.027
1/2	1.81	-1.234	-1.8997	4.127	11.228
3/4	4.10	- .654	-2.141	1.796	11.223

Table 1. $a_n(\alpha)$ for α = 1/4, 1/2 and 3/4.

α	$\int_0^\infty z^{-\alpha-1}\Phi(z)dz$
1/4	2.35
1/2	1.51
3/4	1.07

Table 2. See Equation (12).

Figure 1. Normal traction fields over the contact interval for a parabolic asperity for $\alpha < 1/2$(———); $\alpha = 1/2$(-·-·-·); $\alpha > 1/2$(----).



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α>1/2
α=1/2
α<1/2
S
a₀
-α
1-α
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b₀

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